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A VARIATIONAL PRINCIPLE FOR INCOMPRESSIBLE AND NEARLY-INCOMPRESSIBLE ANISOTROPIC ELASTICITY*

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Abstract—A specialized form of Reissner's variational principle is developed which is suitable for anisotropic incompressible and nearly-incompressible thermoelasticity. The finite element method is used to find solutions in two axisymmetric problems where the material is cylindrically orthotropic and incompressible. The developed variational principle has the feature that the volumetric strain appears in only one equation and errors committed in approximating this equation do not re-enter the stress calculations.

INTRODUCTION

IN ELASTICITY, the minimum potential energy principle is the most common variational statement used. It takes the displacements as unknowns and has as its Euler equations the equilibrium equations in terms of the displacements. The finite element method is a means of finding an approximate solution. It is a numerical solution technique with similarities to the Ritz method that is applied to a variational formulation of a problem. When the finite element method is applied to the minimum potential energy formulation, an approximate displacement field is obtained which is differentiated to find the strains from which the stresses are calculated. This results in fairly coarse answers for the stresses.

An additional difficulty occurs when nearly incompressible materials are considered, as in the solid propellant rocket industry. The equations represented by the minimum potential energy principle contain a coefficient which goes to infinity as Poisson's ratio, ν, approaches one-half. This is reflected in the finite element method by a loss in accuracy in the approximate solution for the displacement field. As a result, the stresses obtained are frequently of little value. It is for this reason that an alternative to the minimum potential energy principle is sought.

Reissner's variational principle is an alternate variational statement of the problem. It utilizes both displacements and stresses as unknowns and has as its Euler equations the equilibrium equations in terms of stresses and the stress–displacement relations. The difficulties encountered in the minimum potential energy principle for incompressible or nearly-incompressible material, do not arise when using Reissner's principle. The two principles are very closely related with Reissner's principle being the canonical form of the minimum potential energy principle, Tonti [1] with the development there ascribed to Fraeijs de Veubeke [2].

F. de Veubeke's derivation of Reissner's principle immediately leads to a modified variational principle which is also suitable for incompressible and nearly-incompressible material. The modified variational principle introduces only a single stress, a pressure.

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The finite element method is used with the modified principle to solve two axisymmetric problems where the material is cylindrically orthotropic and incompressible.

Herrmann [3] has examined this problem for isotropic materials and has presented a modified Reissner's variational principle for use with incompressible and nearly-incompressible isotropic materials. However, the extension to anisotropic and in particular orthotropic materials is not an obvious step.*

REISSNER'S PRINCIPLE AFTER F. DE VEUBEKE

The derivation of Reissner's principle from the minimum potential energy principle is accomplished with a Legendre transformation. Courant and Hilbert [4], page 238, can be consulted regarding this technique. Consider the potential energy functional for anisotropic thermoelasticity. It is

$$\pi_P(u^i) = \iiint_V \left(\tfrac{1}{2}C^{ijrs}u_{(i,j)}u_{(r,s)} - C^{ijrs}u_{(i,j)}\alpha_{rs}\Delta T - \rho f^i u_i\right) \mathrm{d}v - \iint_{S_{\bar{T}}} \bar{T}^i u_i \,\mathrm{d}s \tag{1}$$

where

$C^{ijrs} = C^{jirs} = C^{ijsr} = C^{rsij}$ are the anisotropic elastic constants,
$u_{(i,j)} = \frac{1}{2}(u_{i,j}+u_{j,i})$, are the infinitesimal strains,
α_{rs} are the anisotropic coefficients of thermal expansion,
ΔT is the temperature change,
ρ is the mass density,
f^i are the body forces,
$\bar{T}^i$ are the surface tractions on the boundary $S_{\bar{T}}$,
u_i are the displacements, and
$\bar{u}_i$ are the displacement boundary conditions on $S_{\bar{u}}$.

Indicial notation and the summation convention on repeated upper and lower indices are used. The comma denotes covariant differentiation. Figure 1 shows the problem being considered.

The Euler equations obtained by minimizing π_P on those displacements which are continuous and equal $\bar{u}^i$ on $S_{\bar{u}}$ are the equilibrium equations

$$-(C^{ijrs}u_{(r,s)})_{,j} = \rho f^i - (C^{ijrs}\alpha_{rs}\Delta T)_{,j} \tag{2}$$

and the stress boundary conditions

$$C^{ijrs}u_{(r,s)}n_j = \bar{T}^i + C^{ijrs}\alpha_{rs}\Delta T n_j \qquad \text{on } S_{\bar{T}}, \tag{3}$$

where n_j is the outward normal shown in Fig. 1. Equation (2) represents a set of three simultaneous second order partial differential equations in the three displacements u_i.

Following Tonti [1], a Legendre transformation is made. The generalized stresses, σ^{ij}, are defined as

$$\sigma^{ij} = \frac{\partial L}{\partial u_{(i,j)}} = C^{ijrs}u_{(r,s)} - C^{ijrs}\alpha_{rs}\Delta T, \tag{4}$$

* Since the submission of this manuscript, a paper by Taylor, Pister and Herrmann has appeared, "On a Variational Theorem for Incompressible and Nearly-Incompressible Orthotropic Elasticity," *Int. J. Solids Struct.* **4**, 875–883 (1968). The development there hinges on the introduction of an arbitrary splitting of the elastic compliance tensor which remains throughout the analysis. The present development does not require such a step. Due to this difference, their results are not contained as a special case of the present work.

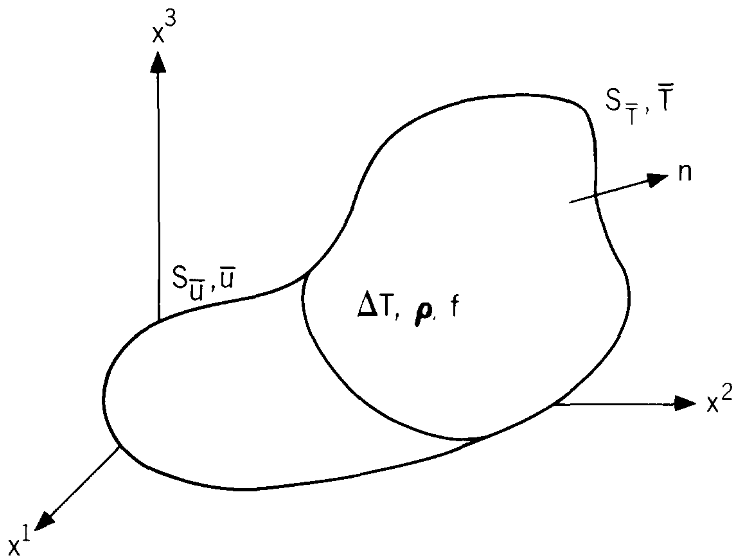

FIG. 1. Problem geometry.

where L is the volume integrand in equation (1). The function H is constructed as

$$H(u_i, \sigma^{ij}) = \sigma^{ij}u_{(i,j)} - L(u_i, u_{(i,j)}). \tag{5}$$

From the function H, the canonical equations are obtained by

$$u_{(i,j)} = \frac{\partial H}{\partial \sigma^{ij}} \quad \text{and} \quad -\sigma^{ij}_{,j} = \frac{\partial H}{\partial u_i}. \tag{6}$$

The function H is written as a function of u_i and σ^{rs} by solving equation (4) for $u_{(r,s)}$ and substituting into equation (5). Equation (4) solved for $u_{(r,s)}$ gives

$$u_{(r,s)} = B_{rsij}\sigma^{ij} + \alpha_{rs}\Delta T, \tag{7}$$

where B_{rsij} is the inverse of C^{ijrs}. The substitution in equation (5) gives

$$H(u_i, \sigma^{rs}) = \tfrac{1}{2}B_{rsij}\sigma^{rs}\sigma^{ij} + \sigma^{rs}\alpha_{rs}\Delta T + \rho f^i u_i, \tag{8}$$

and the canonical equations for this problem are

$$\begin{aligned} -\sigma^{ij}_{,j} &= \rho f^i, \\ u_{(r,s)} &= B_{rsij}\sigma^{ij} + \alpha_{rs}\Delta T. \end{aligned} \tag{9}$$

Reissner's variational principle is obtained by expressing L in terms of the function H from equation (8). The result is

$$\pi_R(u_i, \sigma^{ij}) = \iiint_V (\sigma^{ij}u_{(i,j)} - \tfrac{1}{2}B_{rsij}\sigma^{rs}\sigma^{ij} - \sigma^{rs}\alpha_{rs}\Delta T - \rho f^i u_i)\,\mathrm{d}v - \iint_{S_{\bar{T}}} \bar{T}^i u_i\,\mathrm{d}s. \tag{10}$$

By extremizing the above functional with respect to both the displacements and stress, the equations (9) are obtained.

MODIFIED VARIATIONAL PRINCIPLE

The problems in elasticity which deal with incompressible and nearly incompressible materials can be handled more conveniently by a modified Reissner's variational principle. Reissner's variational principle is adequate for this class of problems but involves a large number of unknown functions. By introducing a single unknown, the pressure p, the difficulties encountered for incompressible or nearly incompressible materials can be avoided.

In essence, the volume strain $u^k_{,k}$ is replaced by the pressure p, and the volume strain is related to the pressure by a constitutive relation. The potential energy functional in terms of displacements, volume strains, and deviatoric strains becomes

$$\pi_P(u_i) = \iiint_V [\tfrac{1}{2}C^{ijrs}u'_{(i,j)}u'_{(r,s)} + \tfrac{1}{3}u^k_{,k}C^{rij}_{r}u'_{(i,j)} + \tfrac{1}{18}C^{rs}_{rs}(u^k_{,k})^2 - C^{ijrs}u'_{(i,j)}\alpha_{rs}\Delta T$$
$$-\tfrac{1}{3}C^{rij}_{r}\alpha_{ij}\Delta T u^k_{,k} - \rho f^i u_i]\,\mathrm{d}v - \iint_{S_T} \bar{T}^i u_i\,\mathrm{d}s, \tag{11}$$

where $u'_{(r,s)} = u_{(r,s)} - \frac{1}{3}u^k_{,k}\delta_{rs}$, the deviatoric strains.

Proceeding as before, the pressure p is defined as

$$p \equiv \frac{\partial L}{\partial u^k_{,k}} = \tfrac{1}{3}C^{rij'}_{r}u'_{(i,j)} + \tfrac{1}{9}C^{rs}_{rs}u^k_{,k} - \tfrac{1}{3}C^{rij}_{r}\alpha_{ij}\Delta T. \tag{12}$$

Solving this equation for the volume strain $u^k_{,k}$ gives

$$u^k_{,k} = 3\left(\frac{3p - C^{kij}_{k}u'_{(i,j)}}{C^{rs}_{rs}}\right) + 3\frac{A^k_k}{C^{rs}_{rs}}, \tag{13}$$

where $A^{rs} = C^{rsij}\alpha_{ij}\Delta T$. The resulting function H is

$$H(u_i, p) = pu^k_{,k} - L(u_i, u'_{(i,j)}, u^k_{,k}) \tag{14}$$
$$= \frac{1}{2C^{rs}_{rs}}(3p - C'^{kij}_{k}u'_{(i,j)})^2 - \tfrac{1}{2}C''^{ijrs}u'_{(i,j)}u'_{(r,s)} + \frac{1}{C^{rs}_{rs}}(3p - C'^{kij}_{k}u'_{(i,j)})A^l_l + A'^{ij}u'_{(i,j)} + \rho f^i u_i.$$

The resulting canonical equations are

$$u^k_{,k} = \frac{\partial H}{\partial p} = 3\left(\frac{3p - C'^{kij}_{k}u'_{(i,j)}}{C^{rs}_{rs}}\right) + 3\frac{A^k_k}{C^{rs}_{rs}},$$
$$-(pg^{ij})_{,j} = \frac{\partial H}{\partial u_i} - \left(\frac{\partial H}{\partial u'_{(i,j)}}\right)_{,j} = \rho f^i + \left(\frac{(3p - C'^{kmn}_{k}u_{(m,n)})}{C^{rs}_{rs}}C'^{kij}_{k}\right)_{,j} \tag{15}$$
$$+ (C^{ijrs}u'_{(r,s)})_{,j} + \left(\frac{A^l_l}{C^{rs}_{rs}}C'^{kij}_{k}\right)_{,j} - A'^{ij}_{,j}.$$

A form of Reissner's variational principle is obtained by expressing L in terms of equation (14). The result is

$$\pi_1(u_i, p) = \iiint_V \Big[p u^k_{,k} - \frac{1}{2C^{rs}_{rs}}(3p - C'^{kij}_k u'_{(i,j)})^2 + \tfrac{1}{2}C''^{ijrs} u'_{(i,j)} u'_{(r,s)} - \frac{A^l_l}{C^{rs}_{rs}}(3p - C'^{kij}_k u'_{(i,j)}) - A'^{ij} u'_{(i,j)} - \rho f^i u_i \Big] \, dv - \iint_{S_{\bar{T}}} \bar{T}^i u_i \, ds. \tag{16}$$

By extremizing the above functional with respect to both the displacements and the pressure, equations (15) are obtained.

This procedure is easily modified to treat a material containing inextensible or nearly-inextensible cords or membranes where the same ill conditioning occurs. For cords, a stress variable is defined using equation (12) based on the extensional strain component aligned with the cords. For a membrane, two stress variables are defined using equation (12) based on the extensional strains in the plane of the membranes. For a "plane" of incompressibility or near-incompressibility, for example, the 1–2 plane, a stress variable is defined using equation (12) based on $u_{1,1} + u_{2,2}$. In each case a variational principle is developed just as above. The analysis in Ref. [7] may also be altered in the same manner. In principie these cords, membranes, and planes need not coincide with coordinate directions; however, the complexity introduced may prevent useful computations.

For isotropic materials, considerable simplification results in both the variational principle and the canonical equations. The variational principle becomes

$$\pi_1(u_i, p) = \iiint_V \Big[p u^k_{,k} - \frac{1}{2}\frac{p^2}{(\lambda + \frac{2}{3}\mu)} + \mu u'_{(i,j)} u'^{(i,j)} - \rho f^i u_i - p 3\alpha \Delta T \Big] \, dv - \iint_{S_{\bar{T}}} \bar{T}^i u_i \, ds. \tag{17}$$

The resulting Euler equations are

$$-p_{,i} - (2\mu u'_{(i,k)} g^{kj})_{,j} = \rho f^i, \qquad u^k_{,k} = \frac{p}{(\lambda + \frac{2}{3}\mu)} + 3\alpha\Delta T. \tag{18}$$

Both the variåtional principle (17) and the Euler equations (18) differ from those presented by Herrmann [3] and used as a basis for finite element computer codes by Pister, Taylor, and Dill [5], and Becker and Brisbane [6].

The differences in formulation can be seen by looking at the equilibrium equations of the variational principle used by Herrmann. The equations are

$$-\frac{3\nu}{1+\nu} p_{,i} - (2\mu u_{(i,k)} g^{kj})_{,j} = \rho f_i - (2\mu\alpha\Delta T)_{,i}.$$

It can be seen that the second term in this equation still contains a volume strain. Not all of the volume strain $u^k_{,k}$ has been replaced by the pressure p. The same is true in the variational principle used by Herrmann. There is, in fact, an infinite number of variational principles between the functional (17) and the variational principle presented by Herrmann The one presented here has the feature that the volumetric strains occur in only one equation. In using the finite element method to obtain an approximate solution, as is done below, errors committed in satisfying equation (15a) will not re-enter the calculation of the stresses.

FINITE ELEMENT SOLUTION

As a test of the practical applicability of the modified Reissner's variational principle, equation (16) has been specialized to orthotropic axisymmetric thermoelasticity, and two problems worked. The existing computer code of Becker and Brisbane [6] was modified to carry out the analysis, Ref. [7]. The cylindrically orthotropic stress–strain–temperature relation used is

$$\begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ e_{rz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{rr}} & \frac{-\nu_{r\theta}}{E_{rr}} & \frac{-\nu_{rz}}{E_{rr}} & 0 \\ \frac{-\nu_{\theta r}}{E_{\theta\theta}} & \frac{1}{E_{\theta\theta}} & \frac{-\nu_{\theta z}}{E_{\theta\theta}} & 0 \\ \frac{-\nu_{zr}}{E_{zz}} & \frac{-\nu_{z\theta}}{E_{zz}} & \frac{1}{E_{zz}} & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{rz}} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{bmatrix} + \begin{bmatrix} \alpha_{rr} \\ \alpha_{\theta\theta} \\ \alpha_{zz} \\ 0 \end{bmatrix} \Delta T, \tag{19}$$

where the following symmetries are required

$$\begin{aligned} \nu_{r\theta}E_{\theta\theta} &= \nu_{\theta r}E_{rr}, \\ \nu_{rz}E_{zz} &= \nu_{zr}E_{rr}, \\ \nu_{\theta z}E_{zz} &= \nu_{z\theta}E_{\theta\theta}. \end{aligned} \tag{20}$$

If the limiting case of an incompressible material is being treated, the following expressions must be satisfied:

$$\begin{aligned} 1-\nu_{r\theta}-\nu_{rz} &= 0, \\ 1-\nu_{\theta r}-\nu_{\theta z} &= 0, \\ 1-\nu_{zr}-\nu_{z\theta} &= 0. \end{aligned} \tag{21}$$

This is a specialization of the condition $B^k_{kij} = 0$ which is obtained from imposing incompressibility on the relation

$$u^k_{,k} = B^k_{kij}\sigma^{ij}. \tag{22}$$

Inverting equation (19) results in

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} C_{rr} & C_{r\theta} & C_{rz} & 0 \\ C_{r\theta} & C_{\theta\theta} & C_{\theta z} & 0 \\ C_{rz} & C_{\theta z} & C_{zz} & 0 \\ 0 & 0 & 0 & G_{rz} \end{bmatrix} \begin{bmatrix} e_{rr}-\alpha_{rr}\Delta T \\ e_{\theta\theta}-\alpha_{\theta\theta}\Delta T \\ e_{zz}-\alpha_{zz}\Delta T \\ 2e_{rz} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned}
C_{rr} &= E_{rr}(1-\nu_{\theta z}\nu_{z\theta})/\Delta,\\
C_{\theta\theta} &= E_{\theta\theta}(1-\nu_{rz}\nu_{zr})/\Delta,\\
C_{zz} &= E_{zz}(1-\nu_{r\theta}\nu_{\theta r})/\Delta,\\
C_{r\theta} &= E_{\theta\theta}(\nu_{r\theta}+\nu_{rz}\nu_{z\theta})/\Delta,\\
C_{rz} &= E_{zz}(\nu_{rz}+\nu_{r\theta}\nu_{\theta z})/\Delta,\\
C_{\theta z} &= E_{zz}(\nu_{\theta z}+\nu_{\theta r}\nu_{rz})/\Delta,\\
\Delta &= 1-\nu_{r\theta}\nu_{\theta z}\nu_{zr}-\nu_{\theta r}\nu_{z\theta}\nu_{rz}-\nu_{r\theta}\nu_{\theta r}-\nu_{\theta z}\nu_{z\theta}-\nu_{zr}\nu_{rz}.
\end{aligned} \tag{24}$$

Using matrix notation, the quadratic form in the deviatoric strains,

$$\tfrac{1}{2}C''^{ijrs}u'_{(i,j)}u'_{(r,s)},$$

becomes

$$\frac{1}{2}\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) \end{bmatrix}^T \begin{bmatrix} C''^{1111} & C''^{1122} & C''^{1133} & 0 \\ C''^{1122} & C''^{2222} & C''^{2233} & 0 \\ C''^{1133} & C''^{2233} & C''^{3333} & 0 \\ 0 & 0 & 0 & 2C''^{1313} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) \end{bmatrix}, \tag{25}$$

where

$$\begin{aligned}
C''^{1111} &= \tfrac{1}{9}(4C_{rr}+C_{\theta\theta}+C_{zz}-4C_{r\theta}-4C_{rz}+2C_{\theta z}),\\
C''^{2222} &= \tfrac{1}{9}(C_{rr}+4C_{\theta\theta}+C_{zz}-4C_{r\theta}+2C_{rz}-4C_{\theta z}),\\
C''^{3333} &= \tfrac{1}{9}(C_{rr}+C_{\theta\theta}+4C_{zz}+2C_{r\theta}-4C_{rz}-4C_{\theta z}),\\
C''^{1122} &= \tfrac{1}{9}(5C_{r\theta}-C_{rz}-C_{\theta z}-2C_{rr}-2C_{\theta\theta}+C_{zz}),\\
C''^{1133} &= \tfrac{1}{9}(5C_{rz}-C_{r\theta}-C_{\theta z}-2C_{rr}-2C_{zz}+C_{\theta\theta}),\\
C''^{2233} &= \tfrac{1}{9}(5C_{\theta z}-C_{r\theta}-C_{rz}-2C_{zz}-2C_{\theta\theta}+C_{rr}),\\
C''^{1313} &= 2G_{rz}.
\end{aligned} \tag{26}$$

Here the computational identity $C^{ijrs'}u'_{(r,s)} \equiv C^{ijrs'}u_{r,s}$ has been used. In matrix notation, the quadratic form

$$pu^k_{,k}$$

becomes

$$\frac{1}{2}\begin{bmatrix} p/E^k_k \\ u^k_{,k} \end{bmatrix}^T \begin{bmatrix} 0 & E^k_k \\ E^k_k & 0 \end{bmatrix} \begin{bmatrix} p/E^k_k \\ u^k_{,k} \end{bmatrix}, \tag{27}$$

$$E^k_k = \tfrac{1}{3}(E_{rr}+E_{\theta\theta}+E_{zz}).$$

Using matrix notation, the quadratic form

$$-\frac{1}{2C_{rs}^{rs}}(3p - C_k'^{kij}u'_{(i,j)})^2$$

becomes

$$-\frac{1}{2C_{rs}^{rs}}\begin{bmatrix} p/E_k^k \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z}\end{bmatrix}^T \begin{bmatrix} 9E_k^{k^2} & -3E_k^k C_k'^{k11} & -3E_k^k C_k'^{k22} & -3E_k^k C_k'^{k33} \\ -3E_k^k C_k'^{k11} & (C_k'^{k11})^2 & C_k'^{k11}C_k'^{k22} & C_k'^{k11}C_k'^{k33} \\ -3E_k^k C_k'^{k22} & C_k'^{k22}C_k'^{k11} & (C_k'^{k22})^2 & C_k'^{k22}C_k'^{k33} \\ -3E_k^k C_k'^{k33} & C_k'^{k33}C_k'^{k11} & C_k'^{k33}C_k'^{k22} & (C_k'^{k33})^2 \end{bmatrix} \begin{bmatrix} p/E_k^k \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z}\end{bmatrix}, \quad (28)$$

where

$$\begin{aligned} C_k'^{k11} &= \tfrac{1}{3}(2C_{rr} - C_{\theta\theta} - C_{zz} + C_{r\theta} + C_{rz} - 2C_{\theta z}), \\ C_k'^{k22} &= \tfrac{1}{3}(+2C_{\theta\theta} - C_{rr} - C_{zz} + C_{r\theta} - 2C_{rz} + C_{\theta z}), \\ C_k'^{k33} &= \tfrac{1}{3}(+2C_{zz} - C_{rr} - C_{\theta\theta} - 2C_{r\theta} + C_{rz} + C_{\theta z}). \end{aligned} \quad (29)$$

The three quadratic forms (25), (27), and (28) are combined to give

$$\frac{1}{2}\begin{bmatrix} p/E_k^k \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z} \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\end{bmatrix}^T \begin{bmatrix} -\frac{9E_k^{k^2}}{C_{rs}^{rs}} & E_k^k\left(1+3\frac{C_k'^{k11}}{C_{rs}^{rs}}\right) & E_k^k\left(1+3\frac{C_k'^{k22}}{C_{rs}^{rs}}\right) & E_k^k\left(1+3\frac{C_k'^{k33}}{C_{rs}^{rs}}\right) & 0 \\ E_k^k\left(1+3\frac{C_k'^{k11}}{C_{rs}^{rs}}\right) & C''^{1111} - (C_k'^{k11})^2 C_{rs}^{rs} & C''^{1122} - C_k'^{k11}C_k'^{k22}/C_{rs}^{rs} & C''^{1133} - C_k'^{k11}C_k'^{k33}/C_{rs}^{rs} & 0 \\ E_k^k\left(1+3\frac{C_k'^{k22}}{C_{rs}^{rs}}\right) & C''^{1122} - C_k'^{k22}C_k'^{k11}/C_{rs}^{rs} & C''^{2222} - (C_k'^{k22})^2/C_{rs}^{rs} & C''^{2233} - C_k'^{k22}C_k'^{k33}/C_{rs}^{rs} & 0 \\ E_k^k\left(1+3\frac{C_k'^{k33}}{C_{rs}^{rs}}\right) & C''^{1133} - C_k'^{k33}C_k'^{k11}/C_{rs}^{rs} & C''^{2233} - C_k'^{k33}C_k'^{k22}/C_{rs}^{rs} & C''^{3333} - (C_k'^{k33})^2/C_{rs}^{rs} & \\ 0 & 0 & 0 & 0 & 2C''^{1313} \end{bmatrix} \begin{bmatrix} p/E_k^k \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z} \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\end{bmatrix}$$

(30

In matrix notation, the linear form

$$-\frac{1}{C^{rs}_{rs}}(3p-C'^{kij}_{k}u'_{(i,j)})C^{lmn}_{l}\alpha_{mn}\Delta T$$

becomes

$$-\begin{bmatrix} p/E^k_k \\ \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) \end{bmatrix}^T \begin{bmatrix} 3E^k_k \\ -C'^{k11}_k \\ -C'^{k22}_k \\ -C'^{k33}_k \\ 0 \end{bmatrix} \left(\frac{C^{l11}_l\alpha_{rr}+C^{l22}_l\alpha_{\theta\theta}+C^{l33}_l\alpha_{zz}}{C^{rs}_{rs}}\right)\Delta T. \tag{31}$$

Using matrix notation, the linear form

$$-A'^{ij}u'_{(i,j)}$$

becomes

$$-\begin{bmatrix} p/E^k_k \\ \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial r}\right) \end{bmatrix}^T \begin{bmatrix} 0 \\ (C'^{1111}\alpha_{rr}+C'^{1122}\alpha_{\theta\theta}+C'^{1133}\alpha_{zz})\Delta T \\ (C'^{1122}\alpha_{rr}+C'^{2222}\alpha_{\theta\theta}+C'^{2233}\alpha_{zz})\Delta T \\ (C'^{1133}\alpha_{rr}+C'^{2233}\alpha_{\theta\theta}+C'^{3333}\alpha_{zz})\Delta T \\ 0 \end{bmatrix}. \tag{32}$$

The remaining linear forms become

$$-\rho f^i u_i = -\begin{bmatrix} u \\ w \\ p/E^k_k \end{bmatrix}^T \begin{bmatrix} \rho f_r \\ \rho f_z \\ 0 \end{bmatrix} \tag{33}$$

and

$$-\bar{T}^i u_i = -\begin{bmatrix} u \\ w \\ p/E^k_k \end{bmatrix}^T \begin{bmatrix} \bar{T}_r \\ \bar{T}_z \\ 0 \end{bmatrix}. \tag{34}$$

In the limit of an incompressible material equations (26), (29) and (32) degenerate into the indeterminate form 0/0 and must be evaluated by taking limits. This process is covered in detail in Ref. [7].

In employing the finite element method to find a solution, a choice in element geometry and an assumption on how the unknowns vary in each element must be made. The computer codes of Pister, Taylor and Dill [5] and Becker and Brisbane [6] use quadrilateral elements obtained by combining four triangular elements as shown in Fig. 2. In both computer codes, a common pressure is taken in all four triangles, giving a constant value to p/E_k^k for the entire quadrilateral. The displacements u and w are assumed to vary linearly in each triangle. These assumptions are retained in the modified computer code of Becker and Brisbane [6].

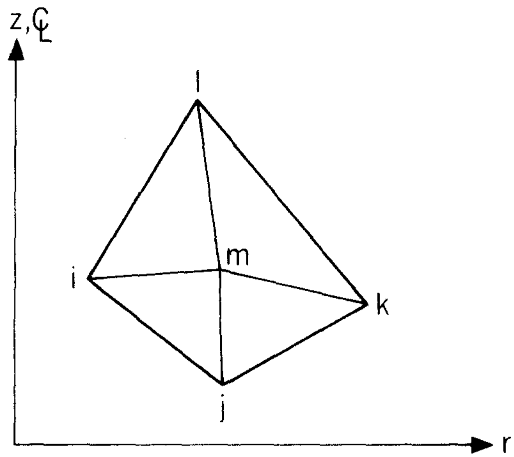

FIG. 2. Quadrilateral finite element.

EXAMPLE SOLUTIONS

The first problem considered is an infinite, hollow, right circular cylinder under internal pressure. Figure 3 shows the geometry used. Analytically, the stresses are given by

$$\begin{aligned} \sigma_{rr} &= \frac{a^2\bar{T}_r}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right), \\ \sigma_{\theta\theta} &= \frac{a^2\bar{T}_r}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right). \end{aligned} \tag{35}$$

Figures 4 and 5 show the behavior of the stresses σ_{rr} and $\sigma_{\theta\theta}$ as the number of elements is increased and demonstrates the behavior of the approximations in the finite element method.

The second problem is a study of the effect of "hoop reinforcements" in an infinite, hollow, right circular cylinder under internal pressure. Figure 3 shows the geometry used. The study was made by taking $E_{rr} = E_{zz} = 1000$ psi and varying $E_{\theta\theta}$ upwards from 1000 psi for an incompressible material. This problem is typical of those encountered in

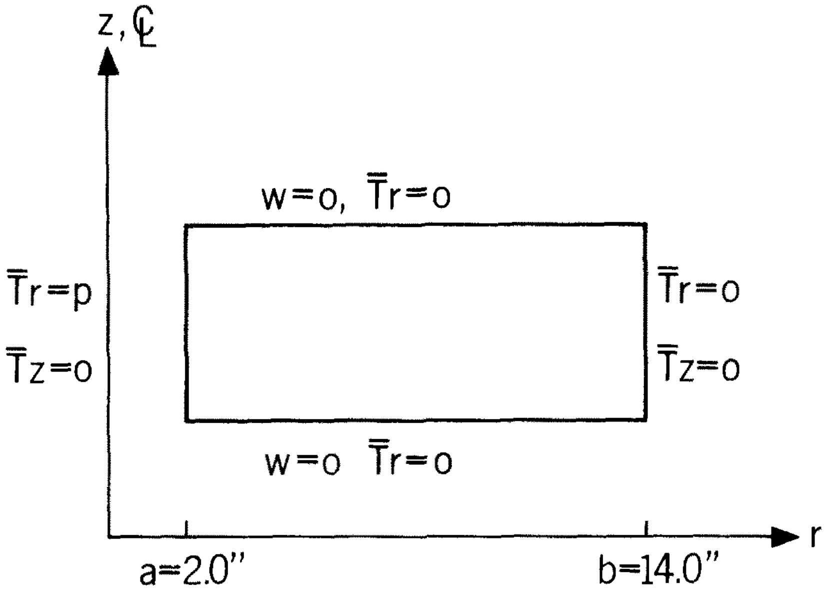

FIG. 3. Hollow infinite cylinder under internal pressure.

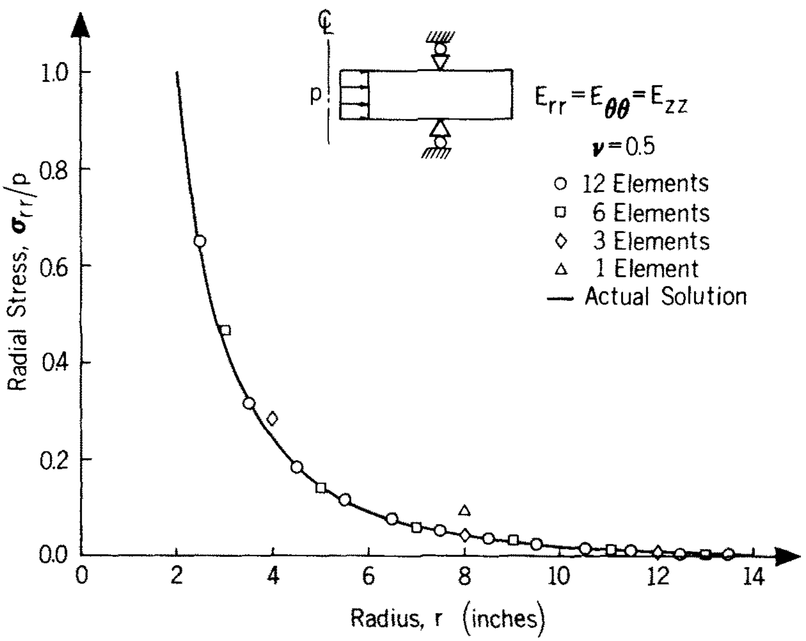

FIG. 4. Convergence of the finite element solution for the radial stresses as the number of elements increases.

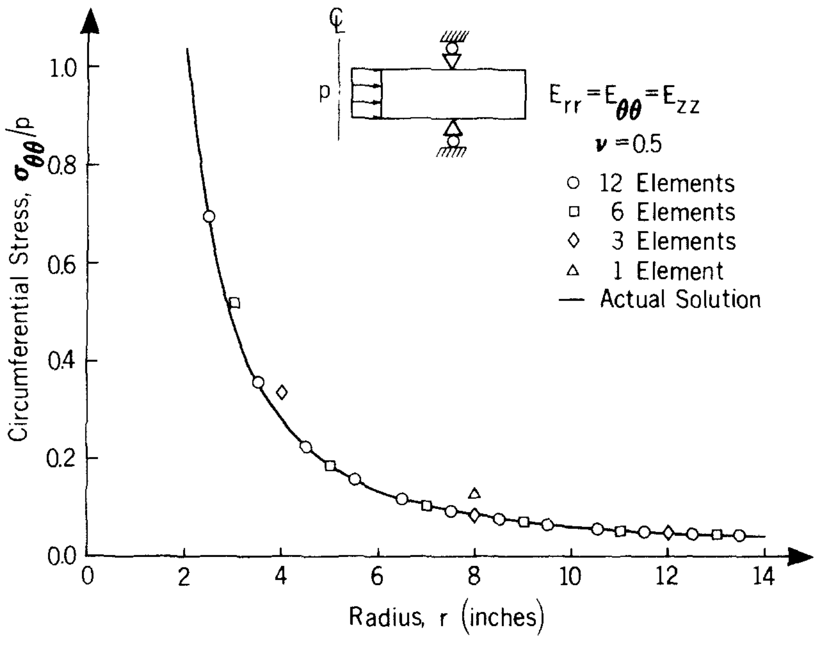

FIG. 5. Convergence of the finite element solution for the circumferential stresses as the number of elements increases.

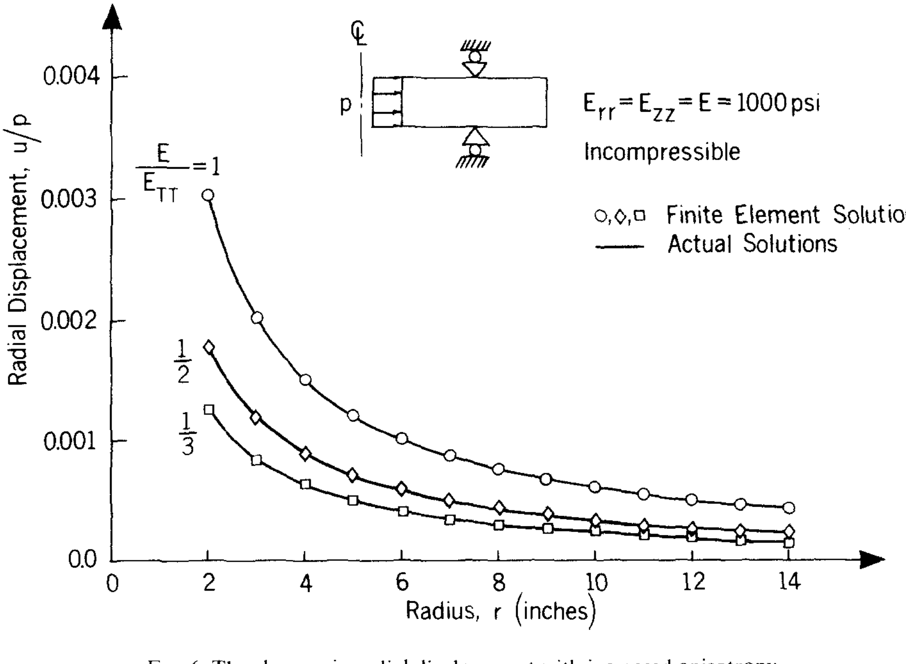

FIG. 6. The changes in radial displacement with increased anisotropy.

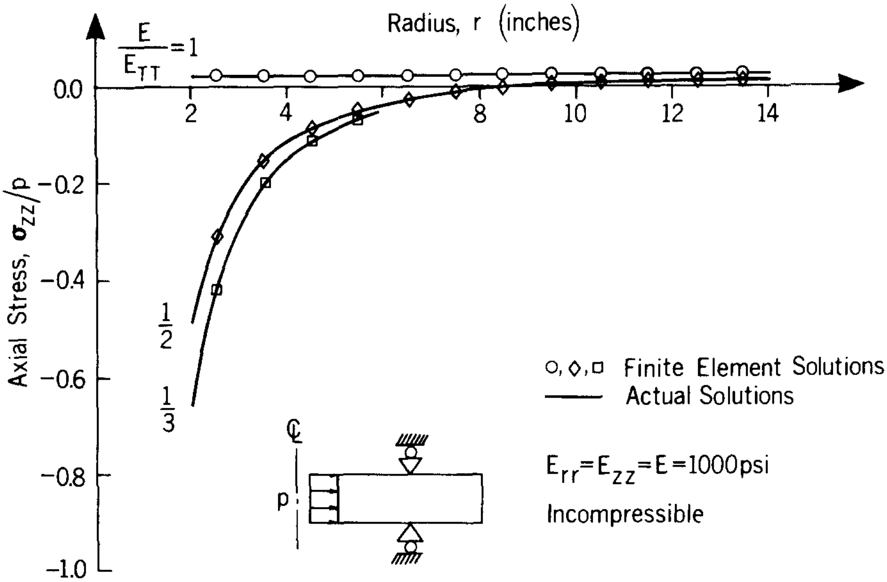

FIG. 7. The changes in axial stress with increased anisotropy.

the analysis of solid propellant rocket engines. Figures 6 and 7 show the results of this study.

The stresses σ_{rr} and $\sigma_{\theta\theta}$ are the same in this second problem as those above since these stresses are independent of the orthotropic material properties. It should be emphasized that the codes written by Becker and Brisbane [6] are capable of solving problems which are much more complex than the simple problems considered above.

CONCLUSIONS

It is shown that incompressible anisotropic thermoelasticity problems can be formulated with the aid of a modified Reissner's variational principle and that the finite element method is a practical means of obtaining results. The calculated stresses in the problem considered are shown to converge well to the analytic solution as the number of elements increases. The effect of "hoop reinforcements" on radial displacements and axial stresses has been calculated for an incompressible material.

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Абстракт—Выводится специфичная форма вариационного принципа Рейсснера, пригодная для анизотропной несжимаемой и поути несжимаемой термоупругости. Используется метод конечного элемента с целью получения решений двух осесимметрических задау, причем материал является цилиндрически ортотропным и несжимаемым. Выведенный вариационный принцип обладает таким свойством, что объемная деформация появляется только в одном уравнении иолремцосмц, вытекаюя щие из приближения этого уравнения, не входят в расчет напряжений.